Recursive Symmetry Coupling: An Axiomatic Framework for Hierarchical Algebraic–Geometric Structures

by
Jan Gaule

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We introduce an axiomatic framework for describing recursively coupled algebraic-geometric structures, whose organization is characterized by recursive symmetry coupling. More generally, the framework may be viewed as describing recursively coupled algebraic-geometric objects whose coupling is mediated by symmetry structures. In contrast to classical foliation theory, where foliations primarily describe geometric decompositions of manifolds, the present framework assigns additional algebraic symmetry structures to families of geometric objects. These structures are not considered independently but correspond recursively through symmetry-preserving coupling operators. The resulting hierarchy exhibits bidirectional organization, where local geometric structures induce algebraic symmetries, while these symmetries recursively reorganize the underlying geometric configuration. We formulate the conceptual principles of the framework, propose a preliminary set of axioms, and discuss possible connections with foliation theory, Lie groupoids, and hierarchical geometric structures.

1 Motivation

The original idea for the presented theory was inspired partly by the theory of foliations. The present framework is motivated by the observation that many geometric structures admit natural algebraic symmetry descriptions. Classical foliation theory captures geometric decompositions of manifolds, whereas the proposed framework aims to describe recursive couplings between geometric and algebraic structures through symmetry-preserving coupling mechanisms.
Geometry induces algebraic symmetries, whereas these symmetries recursively reorganize the underlying geometry. Consequently, neither geometry nor symmetry is considered fundamental in isolation; rather, both emerge through recursive mutual organization.
Classical symmetry theories typically describe invariance of fixed geometric structures. The present framework instead considers geometry and symmetry as mutually evolving entities linked by recursive coupling. Symmetries are considered not as static invariants but as active organizational entities.
As a guiding principle we state that the fundamental elements of the theory are geometric-algebraic objects; their dynamics and hierarchy arise from recursive coupling operators that mediate and continuously reorganize the relationships between these objects.
The statements of the theory presented here can be summarized in the form

Guiding Principle Geometric structures admit algebraic symmetries, while recursively coupled symmetries reorganize the underlying geometry.

Thus, a classical foliation with its holonomy structure is a special case of such a structure.

Motivational Fundamental Statement

Recursive Symmetry Coupling postulates that geometry and algebra are not independent mathematical entities but recursively co-organize one another through symmetry-preserving coupling operators.

2 Fundamental Objects – Core Principles

Definition 1 (Recursive Symmetry Coupling)**.** Recursive symmetry coupling is the process by which one symmetry-coupled structure is transformed into another through a coupling operator while preserving compatibility between the underlying geometric configuration and its associated symmetry object.

Definition 2 (Geometric Configuration)**.** A geometric configuration is a structured collection

$$\mathcal{X} = \{X_i\}_{i \in I} \tag{1}$$

of geometric objects on a topological or differentiable manifold together with their mutual incidence or neighborhood relations.

Definition 3 (Symmetry Objects)**.** A symmetry object

$$\mathcal{G} \tag{2}$$

denotes an algebraic structure associated with a geometric configuration. Typical examples include groups, Lie groups, groupoids, categories, higher categorical structures, and related algebraic objects encoding local or global symmetries.

Definition 4 (Symmetry-Coupled Structure)**.** A symmetry-coupled structure is an ordered pair

$$\Sigma = (\mathcal{G}, \mathcal{X}), \tag{3}$$

consisting of a geometric configuration $\mathcal{X}$ together with an associated symmetry object $\mathcal{G}$.

Definition 5 (Recursive Coupling Operator)**.** A recursive coupling operator

$$\mathcal{C} : \Sigma \to \Sigma' \tag{4}$$

maps one symmetry-coupled structure to another while preserving compatibility between geometry and symmetry.

The above defined entities can be described in a cascading stream through successive organizational levels in the following way:

Geometric Configuration
$\Downarrow$
Symmetry Object
$\Downarrow$
Symmetry-Coupled Structure
$\Downarrow$
Recursive Coupling Operator
$\Downarrow$
New Symmetry-Coupled Structure

3 Recursive Symmetry Coupling

The recursive symmetry coupling is illustrated in the form:

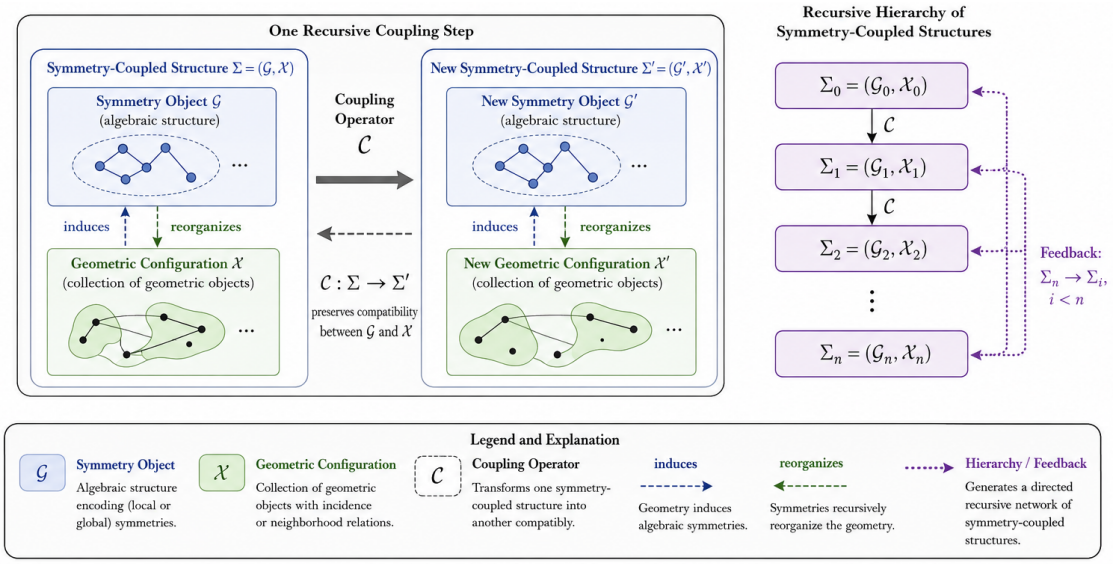

Figure 1: Schematic illustration of recursive symmetry coupling. Each coupling $\mathcal{C}$ transforms a symmetry-coupled structure $\Sigma = (\mathcal{G}, \mathcal{X})$ into a new one $\Sigma' = (\mathcal{G}', \mathcal{X}')$, while preserving compatibility between symmetry and geometry. Repeated application of $\mathcal{C}$ generates a hierarchical network with feedback connections.

4 Foundational Principles

Within the framework of the presented theory we introduce a first set of fundamental axioms.

Axiom 1 (Recursive Symmetry Inheritance)**.** *If two symmetry levels*

$$\mathcal{L}_1 \preceq \mathcal{L}_2 \tag{5}$$

exist, then there exists a morphism with property

$$\Phi : \mathcal{G}(\mathcal{L}_1) \to \mathcal{G}(\mathcal{L}_2) \tag{6}$$

that preserves compatibility between the associated symmetry structures.

Axiom 2 (Hierarchical Organization Principle)**.** *An algebraic structure or symmetrical object organizes general symmerty levels so that an arrangement according to*

$$\Sigma_0 \xrightarrow{\mathcal{C}} \Sigma_1 \xrightarrow{\mathcal{C}} \Sigma_2 \xrightarrow{\mathcal{C}} \cdots \tag{7}$$

with

$$\Sigma_n \to \Sigma_i\,; \quad (i < n) \tag{8}$$

emerges, where every symmetry level consists of geometric objects together with their associated symmetry structures. This recursive symmetry hierarchy is a directed recursive network of symmetry-coupled structures.

Axiom 3 (Recursive Coupling Principle)**.** *Let* $\Sigma = (\mathcal{G}, \mathcal{X})$ *denote a symmetry-coupled structure consisting of a geometric configuration* $\mathcal{X}$ *together with an associated algebraic symmetry object* $\mathcal{G}$*. There exists a coupling operator implementing recursive coupling via*

$$\mathcal{C} : \Sigma \to \Sigma'\,, \tag{9}$$

which recursively generates a new symmetry-coupled structure. Repeated application of $\mathcal{C}$ *gives rise to a hierarchy of recursively coupled structures.*

Axiom 4 (Bidirectional Symmetry Principle)**.** *Let*

$$\Sigma = (\mathcal{G}, \mathcal{X}) \tag{10}$$

be a symmetry-coupled structure.
Any admissible modification of the geometric configuration induces a corresponding transformation of the associated symmetry object. Conversely, every admissible transformation of the symmetry object induces a compatible reorganization of the underlying geometric configuration.
Neither direction is regarded as mathematically fundamental.

5 Outlook

The present work should be understood as the formulation of an axiomatic framework rather than a complete mathematical theory. Its purpose is to identify the fundamental principles of recursive symmetry coupling and to provide a basis for future developments, examples, and applications. It is intentionally formulated at a conceptual level in order to encompass a broad class of existing and future algebraic-geometric theories as special realizations. Possible directions for future work include explicit realizations of the proposed framework in foliation theory, Lie groupoids, higher symmetry structures, and related areas of differential geometry.